Generalized eigenfunctions of relativistic Schrödinger operators in two dimensions

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Abstract

Generalized eigenfunctions of the two-dimensional relativistic Schrödinger operator $H = \sqrt{-\Delta} + V(x)$ with $|V(x)| \le C\langle x\rangle^{-\sigma}$, $\sigma > 3/2$, are considered. We compute the integral kernels of the boundary values $R_0^{\pm}(\lambda) = (\sqrt{-\Delta} - (\lambda \pm i0))^{-1}$, and prove that the generalized eigenfunctions $\varphi^{\pm}(x,k)$ are bounded on $R_x^2 \times \{k \,|\, a \le |k| \le b\}$, where $[a,b] \subset (0,\infty)\backslash\sigma_p(H)$, and $\sigma_p(H)$ is the set of eigenvalues of H. With this fact and the completeness of the wave operators, we establish the eigenfunction expansion for the absolutely continuous subspace for H. Finally, we show that each generalized eigenfunction is asymptotically equal to a sum of a plane wave and a spherical wave under the assumption that $\sigma > 2$.

1 Introduction

Generalized eigenfunctions for Schrödinger operators $-\Delta + V(x)$ on $\mathbb{R}^n$ are now well understood at least in the framework of simple scattering; see for example Agmon[1], Ikebe[5] and Kato and Kuroda[7]. In the pseudo-relativistic regime, one can replace the Schrödinger operators with relativistic Schrödinger operators $\sqrt{-\Delta + m} + V(x)$. Here m is the mass of the particle, and it could be zero. In this case, we deal with the operators of the form $\sqrt{-\Delta} + V(x)$.

This paper is a continuation of our previous paper Wei [19], where the odd-dimensional relativistic Schrödinger operators $\sqrt{-\Delta} + V(x)$ were considered and substantial generalizations of the results by Umeda [16], [17], who only

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dealt with the three-dimensional case, were accomplished. In the present paper, we shall deal with the two-dimensional case:

$$H = H_0 + V(x), \quad H_0 = \sqrt{-\Delta}, \quad x \in \mathbb{R}^2. \tag{1.1}$$

Our aim here is to establish all the same results as in [16], [17] and [19]. For this reason and for the sake of simplicity, we shall use the same notation as in [19].

We now roughly recall the discussions demonstrated in our previous works [16], [17] and [19] for the reader's convenience. We first defined the generalized eigenfunctions $\varphi^{\pm}(x,k)$ with the aid of the limiting absorption principle for the relativistic Schrödinger operators. We next proved that the generalized eigenfuctions are bounded on the set $(x,k) \in \mathbb{R}^n \times \{k \in \mathbb{R}^n \mid a \le |k| \le b\}$ for $[a,b] \subset (0,\infty) \backslash \sigma_p(\sqrt{-\Delta} + V(x))$, where $n = 3, 5, 7, \cdots$, and $\sigma_p(\sqrt{-\Delta} + V(x))$ denotes the point spectrum. Then we showed the asymptotic completeness of the wave operators by the Enss method (cf. [3, 6]), and obtained the eigenfunction expansions for the absolutely continuous subspace for $\sqrt{-\Delta} + V(x)$. In the three dimensional case, we gave estimates on the differences between the generalized eigenfunctions and the plane waves. Moreover we showed that the generalized eigenfunctions are asymptotically equal to the sum of plane waves and spherical waves. It should be remarked that once we have the boundedness of the generalized eigenfunctions, we are able to establish the completeness of the generalized eigenfunctions for the absolutely continuous subspace. (See [19]. Also see [8, 9].)

Our basic assumption is as follows.

Assumption: $V(x)$ is a real-valued measurable function on $\mathbb{R}^2$ satisfying

$$|V(x)| \le C\langle x\rangle^{-\sigma}, \quad \sigma > 3/2. \tag{1.2}$$

Under the assumption (1.2), it is obvious that $V = V(x)\times$ is a bounded selfadjoint operator in $L^2(\mathbb{R}^2)$, and that $H = H_0 + V$ defines a selfadjoint operator in $L^2(\mathbb{R}^2)$, whose domain is $H^1(\mathbb{R}^2)$, the Sobolev space of order one. Moreover H is essentially selfadjoint on $C_0^\infty(\mathbb{R}^2)$ (see [17, sections 2 and 7]). Note that

$$\sigma_e(H) = \sigma_e(H_0),$$

where $\sigma_e(H)$ and $\sigma_e(H_0)$ denote the essential spectrum of H and H_0 respectively. This fact follows from Reed and Simon [12, p.113, Corollary 2], since V is relatively compact with respect to H_0. Also, note that the essential spectrum of H_0 coincides with the spectrum of H_0: $\sigma_e(H_0) = \sigma(H_0) = [0, +\infty)$.

The main idea in this paper is essentially the same as in [17] and [19]. Thus we basically follow the same line as in [17] and [19]. Namely, we first prove

the boundedness of the generalized eigenfunctions, and then we establish the eigenfunction expansion, and finally we examine asymptotic behaviors of the generalized eigenfunctions at infinity.

However, we should like to emphasize that some difficulties specific to the two-dimensional case arise. One should recall that there are significant differences between the two-dimensional wave equation and the three-dimensional one in their treatments. We find that a similar phenomenon is also observed in the treatments of relativistic Schrödinger operators.

In the odd-dimensional case, the integral kernel of the resolvent of the operator $\sqrt{-\Delta}$ is expressed in terms of trigonometric functions, and the cosine and sin integral functions (see [17] and [19]). On the other hand, we encounter the Bessel function, Neumann function and the Struve function in the integral kernel of the resolvent of $\sqrt{-\Delta}$ in the two-dimensional case. This difference makes the analysis of the resolvent of $\sqrt{-\Delta}$ more difficult in the two-dimensional case.

In fact, when we deal with the boundary values of the the resolvent $R_0(z)$ to define the generalized eigenfunctions of H, we are obliged to examine the boundary values of all of the Bessel function, the Neumann function and the Struve function on the positive half line $[0, +\infty)$. It is surprising that a suitable combination of these special functions on the positive half line exhibits a simple form of an exponential function at infinity. This fact enables us to show that the generalized eigenfunctions of relativistic Schrödinger operators in the two-dimensional case too are asymptotically equal to superpositions of plane waves and spherical waves at infinity .

We would like to mention a technicality. In showing the boundedness of generalized eigenfunctions in section 4, we need to handle the Riesz potential on $\mathbb{R}^2$. We shall show that for functions in a certain class the Riesz potential defines bounded functions. We believe that this fact, as well as our technique, is interesting in its own right. The key for this fact is the estimate (4.14), which is based on Lemma 4.5.

We expect that the discussions on the generalized eigenfunctions in the $2m$ dimensional case ($m \geq 2$) would become more complicated, and will be discussed elsewhere ([20]).

The plan of the paper In section 2, we define generalized eigenfunctions of H. In section 3, we compute the resolvent kernel of H_0. Section 4 is devoted to prove the boundedness of the generalized eigenfunctions. In section 5, we deal with the completeness of the generalized eigenfunctions for the absolutely continuous subspace for H. Finally, in section 6, we examine the asymptotic behaviors of the generalized eigenfunctions at infinity. In appendix, we include two inequalities which are used repeatedly in the present paper, and summarize some basic properties of the Bessel, Neumann and Struve functions for the reader's convenience.

2 Generalized eigenfuctions

By $R(z)$ and $R_0(z)$, we mean the resolvents of H and H_0 respectively:

$$R(z) := (H-z)^{-1}, \;\; R_0(z) := (H_0-z)^{-1}. \tag{2.1}$$

The task of this section is to construct generalized eigenfunctions $\varphi^{\pm}(x,\,k)$ of $\sqrt{-\Delta}+V(x)$ (see Theorem 2.3 below), and show that they satisfy

$$\varphi^{\pm}(x,k) = \varphi_0(x,k) - R_0^{\mp}(|k|)V\varphi^{\pm}(x,k), \tag{2.2}$$

where $R_0^{\pm}(z)$ denotes the extended resolvents of H_0 (cf. Theorem 2.1 below) and $\varphi_0(x,k)$ denotes the plane wave

$$\varphi_0(x,k) = e^{ix\cdot k}. \tag{2.3}$$

As we shall see in Theorem 3.2 in section 3, the extended resolvents $R_0^{\pm}(\lambda)$ have the integral kernels $g_\lambda^{\pm}(x-y)$. Since we have

$$g_\lambda^{\pm}(x) \approx \left(\frac{\lambda}{\pi}\right)^{1/2}(1\mp i)\frac{e^{\mp i(\lambda|x|-\pi/4)}}{|x|^{1/2}}$$

as $|x|\to\infty$ (see (3.19) below), it is justified to call (2.2) the Lippman-Schwinger type integral equations.

The discussions in this section are based on the results by Ben-Artzi and Nemirovski [2, sections 2 and 4]. Since their results are formulated in a general setting, we reproduce them in the context of the present paper.

Theorem 2.1 (Ben-Artzi and Nemirovski [2]) *Let $s>1/2$. Then*
(1) *For any $\lambda>0$, there exist the limits $R_0^{\pm}(\lambda) = \lim_{\mu\downarrow 0} R_0(\lambda\pm i\mu)$ in $\boldsymbol{B}(L^{2,s}, H^{1,-s})$.*
(2) *The operator-valued functions $R_0^{\pm}(z)$ defined by*

$$R_0^{\pm}(z) = \begin{cases} R_0(z) & \text{if} \quad z\in\mathbb{C}^{\pm} \\ R_0^{\pm}(\lambda) & \text{if} \quad z=\lambda>0 \end{cases}$$

are $\boldsymbol{B}(L^{2,s}, H^{1,-s})$-valued continuous functions, where $\mathbb{C}^+$ and $\mathbb{C}^-$ are the upper and the lower half-planes respectively: $\mathbb{C}^{\pm} = \{z\in\mathbb{C} \mid \pm\,\mathrm{Im}\; z>0\}$.

Theorem 2.2 (Ben-Artzi and Nemirovski [2]) *Let $s>1/2$ and $\sigma>1$. Then*
(1) *The continuous spectrum $\sigma_c(H)=[0,\infty)$ is absolutely continuous, except possibly for a discrete set of embedded eigenvalues $\sigma_p(H)\cap(0,\infty)$, which can accumulate only at 0 and ∞.*
(2) *For any $\lambda\in(0,\infty)\backslash\sigma_p(H)$, there exist the limits*

$$R^{\pm}(\lambda) = \lim_{\mu\downarrow 0} R(\lambda\pm i\mu) \quad \text{in} \quad \boldsymbol{B}(L^{2,s}, H^{1,-s}).$$

(3) The functions $R^{\pm}(z)$ defined by

$$R^{\pm}(z) = \begin{cases} R(z) & \text{if} \quad z \in \mathbb{C}^{\pm} \\ R^{\pm}(\lambda) & \text{if} \quad z = \lambda \in (0, \infty)\backslash\sigma_p(H) \end{cases}$$

are $\boldsymbol{B}(L^{2,s}, H^{1,-s})$-valued continuous.

Now we can follow the arguments in our previous papers [17, section 8] and [19, section 1] with a few of obvious changes, and obtain the following two theorems.

Theorem 2.3 ([17], [19]) *If $|k| \in (0, \infty)\backslash\sigma_p(H)$, then the eigenfunctions defined by*

$$\varphi^{\pm}(x, k) = \varphi_0(x, y) - R^{\mp}(|k|)\{V(\cdot)\varphi_0(\cdot, k)\}(x), \tag{2.4}$$

satisfy the equation

$$(\sqrt{-\Delta_x} + V(x))u = |k|u \quad in \quad \mathcal{S}'(\mathbb{R}^2_x).$$

Theorem 2.4 ([17], [19]) *If $|k| \in (0, \infty)\backslash\sigma_p(H)$ and $1 < s < \sigma - 1/2$, then we have*

$$\varphi^{\pm}(x, k) = \varphi_0(x, k) - R_0^{\mp}(|k|)\{V(\cdot)\varphi^{\pm}(\cdot, k)\}(x) \quad in \quad L^{2,-s}(\mathbb{R}^2).$$

3 The integral kernels of the resolvents of H_0

This section is devoted to computing the kernel $g_z(x-y)$ of the resolvent $R_0(z)$. What we shall need in the later sections is the limit $g^{\pm}_{\lambda}(x)$ of the function $g_{\lambda \pm i\mu}(x)$ as $\mu \downarrow 0$, where $\lambda > 0$. Then we derive a few inequalities for the extended resolvent $R^{\pm}_0(\lambda)$, using some estimates of the functions $g^{\pm}_{\lambda}(x)$.

We first need to introduce the following functions.

$$M_z(x) = \frac{1}{2}\Big\{\mathbf{H}_0(-|x|z) - N_0(-|x|z)\Big\}, \quad z \in \mathbb{C} \setminus [0, +\infty) \tag{3.1}$$

$$m^{\pm}_{\lambda}(x) = -\frac{1}{2}\Big\{\mathbf{H}_0(|x|\lambda) + N_0(|x|\lambda) \pm 2iJ_0(|x|\lambda)\Big\}, \quad \lambda > 0. \tag{3.2}$$

Here $\mathbf{H}_0(z)$ is the Struve function (cf. [11, p.227, p.228], [18, p.328]), $N_0(z)$ the Neumann function (cf. [11, p.145, p.146], [18, p.62, p.64]; the Neumann function

is denoted by $Y_0(z)$ in [18]) and $J_0(z)$ the Bessel function (cf. [11, p.145, p.146], [18, p.40]):

$$\mathbf{H}_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+1}}{\{\Gamma(k+3/2)\}^2}, \tag{3.3}$$

$$N_0(z) = \frac{2}{\pi} J_0(z)(\gamma + \log(z/2)) - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k (z/2)^{2k} \sum_{m=1}^{k} \frac{1}{m}}{(k!)^2} \tag{3.4}$$

(γ the Euler constant).

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{(n!)^2}. \tag{3.5}$$

Note that the Struve function $\mathbf{H}_0(z)$ and the Bessel function $J_0(z)$ are both entire functions. Also note that the Neumann function $N_0(z)$ is a many-valued function with a logarithmic branch-point at $z = 0$. Here we choose the principal branch, i.e. $|\mathrm{Im}\ \log z| < \pi$ for $z \in \mathbb{C} \setminus (-\infty, 0]$.

The resolvent kernel of H_0 is given as follows.

Theorem 3.1 *If $z \in \mathbb{C} \setminus [0, +\infty)$, then*

$$R_0(z)u = G_z u$$

for all $u \in C_0^\infty(\mathbb{R}^2)$, where

$$G_z u(x) = \int_{\mathbb{R}^2} g_z(x-y)u(y)dy, \tag{3.6}$$

$$g_z(x) = \frac{1}{\pi |x|} + z M_z(x). \tag{3.7}$$

Proof. We follow the same line as in [17, senction 2] and [19, section 2], and we only give the sketch of the proof.

We start with the Poison kernel

$$P_t(x) = \frac{t}{\pi(t^2 + |x|^2)^{3/2}},$$

and the fact that $e^{-tH_0}u = P_t * u$ for $t > 0$ and $u \in L^2(\mathbb{R}^2)$. Then we appeal to the fact that

$$R_0(z) = \int_0^{+\infty} e^{tz} e^{-tH_0} dt, \quad \mathrm{Re}\ z < 0.$$

For all $u,\, v \in C_0^\infty(\mathbb{R}^2)$ we have

$$
\begin{aligned}
&(R_0(z)u,\, v)_{L^2} \\
&= \int_0^{+\infty} e^{tz}\,(e^{-tH_0}u,\, v)_{L^2}\,dt \\
&= \int_{\mathbb{R}^2}\Big\{\int_{\mathbb{R}^2}\Big(\int_0^{+\infty} e^{tz}\frac{t}{\pi(t^2+|x-y|^2)^{3/2}}\,dt\Big)\,u(y)\,dy\Big\}\overline{v(x)}\,dx
\end{aligned} \tag{3.8}
$$

for z with Re $z < 0$, where we have made a change of order of integration. (Note that the integral in (3.8) is absolutely convergent. See the proof of [19, Lemma 2.2], which is valid in any dimension $n \geq 2$.) It is evident that the integration with respect t in (3.8) gives the integral kernel of $R_0(z)$ if Re $z < 0$. For this reason we make the following computation:

$$
\begin{aligned}
&\int_0^\infty e^{tz}\frac{t}{\pi(t^2+|x|^2)^{3/2}}dt \\
&= \Big[-\frac{e^{tz}}{\pi\sqrt{t^2+|x|^2}}\Big]_0^\infty + z\int_0^\infty \frac{e^{tz}}{\pi\sqrt{t^2+|x|^2}}dt \\
&= \frac{1}{\pi|x|} + zM_z(x) \\
&= g_z(x)
\end{aligned} \tag{3.9}
$$

if Re $z < 0$. Here we have used the formula

$$
\int_0^\infty \frac{e^{tz}}{\pi\sqrt{t^2+|x|^2}}dt = \frac{1}{2}\Big\{\mathbf{H}_0(-|x|z) - N_0(-|x|z)\Big\}
$$

(cf. [4, p.138], [10, p.289]; note that the Neumann function is denoted by $Y_0(z)$ in [4]). Summing up, we have shown that

$$
(R_0(z)u,\, v)_{L^2} = (G_z u,\, v)_{L^2} \tag{3.10}
$$

for all $u,\, v \in C_0^\infty(\mathbb{R}^2)$ when Re $z < 0$. Since both sides of (3.10) are holomorphic functions of z on $\mathbb{C} \setminus [0,\, +\infty]$, we get the conclusion of the theorem. $\square$

In the proof of Theorem 3.2 below, we shall need the following estimates (Appendix B): For $\rho > 0$

$$
|J_0(\rho)| \leq \text{const.}\begin{cases} 1 & \text{if } 0 < \rho \leq 1, \\ \rho^{-1/2} & \text{if } \rho \geq 1, \end{cases}
$$

$$
|N_0(\rho)| \leq \text{const.}\begin{cases} |\log\rho| & \text{if } 0 < \rho \leq 1, \\ \rho^{-1/2} & \text{if } \rho \geq 1, \end{cases}
$$

$$
|\mathbf{H}_0(\rho)| \leq \text{const.}\begin{cases} \rho & \text{if } 0 < \rho \leq 1, \\ \rho^{-1/2} & \text{if } \rho \geq 1. \end{cases}
$$

Since $|\log\rho| \le \text{const.}\rho^{-1/2} \quad (0 < \rho \le 1)$, we see that

$$|m_\lambda^\pm(x)| \le \text{const.}(|x|\lambda)^{-1/2}. \tag{3.11}$$

Theorem 3.2 *If $\lambda > 0$, then*

$$R_0^\pm(\lambda)u = G_\lambda^\pm u$$

for all $u \in C_0^\infty(\mathbb{R}^2)$, where

$$\begin{aligned} G_\lambda^\pm u(x) &= \int_{\mathbb{R}^2} g_\lambda^\pm(x-y)u(y)dy, \\ g_\lambda^\pm(x) &= \frac{1}{\pi|x|} + \lambda m_\lambda^\pm(x). \end{aligned} \tag{3.12}$$

Proof. Again we follow the same line as in [17, senction 4] and [19, section 2], and we only give the sketch of the proof.

It follows from Theorem 3.1 that

$$(R_0(\lambda \pm i\mu)u,\, v)_{L^2} = (G_{\lambda\pm i\mu}u,\, v)_{L^2} \tag{3.13}$$

for all $u,\, v \in C_0^\infty(\mathbb{R}^2)$ whenever $\lambda > 0$, $\mu > 0$. Regarding $R_0(\lambda \pm i\mu)u \in L^{2,-s}$ and $v \in L^{2,s}$ for some $s > 1/2$, we apply Theorem 2.1 to the left-hand side of (3.13), and see that

$$\lim_{\mu\downarrow 0}(R_0(\lambda \pm i\mu)u,\, v)_{L^2} = (R_0^\pm u,\, v)_{-s,s}. \tag{3.14}$$

Here $(\cdot,\, \cdot)_{-s,s}$ denotes the anti-duality bracket or the pairing between $L^{2,-s}$ and $L^{2,s}$. To examine the limit of the right-hand side of (3.13), we see that

$$\begin{aligned} &\lim_{\mu\downarrow 0} \mathbf{H}_0(-|x|(\lambda \pm i\mu)) = -\mathbf{H}_0(|x|\lambda), \\ &\lim_{\mu\downarrow 0} J_0(-|x|(\lambda \pm i\mu)) = J_0(|x|\lambda), \\ &\lim_{\mu\downarrow 0} N_0(-|x|(\lambda \pm i\mu)) = N_0(|x|\lambda) \pm 2iJ_0(|x|\lambda). \end{aligned}$$

These facts, together with (3.1), (3.2), (3.7) and (3.12), show that

$$\lim_{\mu\downarrow 0} g_{\lambda\pm i\mu}(x) = \frac{1}{\pi|x|} + \lambda m_\lambda^\pm(x) = g_\lambda^\pm(x). \tag{3.15}$$

By virtue of (3.11), we can apply the Lebesgue dominated convergence theorem to the right-hand side of (3.13), and we get

$$\lim_{\mu\downarrow 0}(G_{\lambda\pm i\mu}u,\, v)_{L^2} = \iint_{\mathbb{R}^4} g_\lambda^\pm(x-y)u(y)\overline{v(y)}\,dx\,dy. \tag{3.16}$$

Combining (3.14) and (3.16), we get the conclusion of the theorem. □

It follows from Theorem 3.2 that the integral operator $G_\lambda^\pm$ can be extended to bounded operators from $L^{2,s}$ to $H^{1,-s}$ for $s > 1/2$.

We shall show the boundedness of the generalized eigenfunctions $\varphi^\pm(x,k)$ in section 4, where we shall use the following integral operators:

$$T_j u(x) := \int_{\mathbb{R}^2} |x-y|^{-j} u(y)dy, \quad j = 1,\, 1/2 \tag{3.17}$$

Recall that these integral operators are actually Riesz potentials up to constants.

Following lemma is a direct consequence of Theorem 3.2, (3.11)and (3.17).

Lemma 3.3 *Let $s > 1/2$. If $[a,\, b] \subset (0,\, +\infty)$. then there exist a positive constant C_{ab} such that*

$$|R_0^\pm(\lambda)u(x)| \le \frac{1}{\pi}\big|T_1 u(x)\big| + C_{ab}\big(T_{1/2}|u|\big)(x) \tag{3.18}$$

for all $u \in L^{2,s}$ and all $\lambda \in [a,\, b]$.

We prepare one more lemma for a later purpose.

Lemma 3.4 *For each $\lambda > 0$ we have*

$$g_\lambda^\pm(x) = \left(\frac{\lambda}{\pi}\right)^{1/2}(1 \mp i)\frac{e^{\mp i\lambda|x|}}{|x|^{1/2}} + O(|x|^{-1}) \tag{3.19}$$

as $|x| \to \infty$.

Proof. Apply Lemmas B.1 and B.2 in the appendix to (3.2). □

4 Boundedness of the generalized eigenfunctions

In this section, we shall discuss the boundedness of the generalized eigenfuctions $\varphi^\pm(x,k)$ defined in Theorem 2.3. Following our previous papers [17] and [19], we shall need a restriction on k. Namely, we assume that k satisfies the following inequality:

$$a \le |k| \le b, \tag{4.1}$$

where $[a,b] \subset (0,\infty)\backslash\sigma_p(H)$. As we have seen in Theorem 2.4, the generalized eigenfuction $\varphi^\pm(x,k)$ satisfies the equation

$$\varphi^\pm(x,k) = \varphi_0(x,k) - R_0^\mp(|k|)\{V(\cdot)\varphi^\pm(\cdot,k)\}(x). \tag{4.2}$$

In section 3, we have shown that $R_0^{\mp}(|k|)$ are integral operators, and investigated properties of the integral kernels.

We are now in a position to state the main theorem in this section, which is stated as follows.

Theorem 4.1 *Let $[a,b] \subset (0,\infty)\backslash\sigma_p(H)$. There exists a constant C_{ab} such that generalized eigenfunctions defined by (2.4) satisfy*

$$|\varphi^{\pm}(x,k)| \leq C_{ab} \tag{4.3}$$

for all $(x,k) \in \mathbb{R}^2 \times \{\, k \,|\, a \leq |k| \leq b \,\}$.

Before proving Theorem 4.1, we have to prepare a few lemmas. With application of Theorem 3.2 in mind, we shall show that $V(x)\varphi^{\pm}(x,k)$ belongs to $L^{2,s}(\mathbb{R}^2_x)$ provided that $1/2 < s < \sigma - 1$. To this end, we put

$$\psi^{\pm}(x,k) = V(x)\varphi^{\pm}(x,k). \tag{4.4}$$

Lemma 4.2 *If $1/2 < s < \sigma - 1$, then $\psi^{\pm}(x,k)$ are $L^{2,s}(\mathbb{R}^2_x)$–valued continuous functions on $\left\{\, k \,\middle|\, |k| \in (0,\infty)\backslash\sigma_p(H) \right\}$.*

Proof. Since we have [19, Lemma 1.1] with $n = 2$, we can imitate the arguments in [17, Lemmas 9.2 and 9.3], and see that for any $t > 1$, $\psi^{\pm}(x,k)$ are $L^{2,\sigma-t}(\mathbb{R}^2_x)$-valued continuous functions on $\left\{\, k \,\middle|\, |k| \in (0,\infty)\backslash\sigma_p(H) \right\}$. For $s \in (1/2,\, \sigma - 1)$, we put $t := \sigma - s$. Then $t > 1$, and hence we get the lemma. $\square$

Lemma 4.3 *If $4/3 < r < 2$, then $\psi^{\pm}(x,k)$ are $L^r(\mathbb{R}^2_x)$–valued continuous fuctions on $\left\{\, k \,\middle|\, |k| \in (0,\infty)\backslash\sigma_p(H) \right\}$.*

Proof. Applying the Hölder inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} |\psi^{\pm}(x,k)|^r dx \leq & \left\{ \int_{\mathbb{R}^2} \left(\langle x \rangle^{-r/2} \right)^{2/(2-r)} dx \right\}^{(2-r)/2} \\
& \times \left\{ \int_{\mathbb{R}^2} \left(\langle x \rangle^{r/2} |\psi^{\pm}(x,k)|^r \right)^{2/r} dx \right\}^{r/2} \\
= & \left\{ \int_{\mathbb{R}^2} \langle x \rangle^{-r/(2-r)} dx \right\}^{(2-r)/2} \left\{ \int_{\mathbb{R}^2} \langle x \rangle \, |\psi^{\pm}(x,k)|^2 dx \right\}^{r/2} \\
= & C_r \left(\|\psi^{\pm}\|_{L^{2,1/2}} \right)^{r/2} \\
\leq & C_r \left(\|\psi^{\pm}\|_{L^{2,s}} \right)^{r/2} < \infty,
\end{aligned}$$

where C_r is a constant depending only on r and $s \in (1/2,\, \sigma - 1)$. Here we have used the fact that $r/(2-r) > 2$ if and only if $4/3 < r < 2$. Lemma 4.2, together

with this inequality, implies that $\psi^{\pm}(x,k)$ belongs to $L^r(\mathbb{R}^2_x)$ if $4/3 < r < 2$. Moreover, by using a similar argument, one can easily show that $\psi^{\pm}(x,k)$ are $L^r(\mathbb{R}^2_x)$-valued continuous functions on $\{\, k \,\big|\, |k| \in (0,\infty)\backslash\sigma_p(H)\big\}$. $\square$

For the sake of simplicity, we shall apply Lemma 4.3 with $r = 16/9$:

Lemma 4.4 *$\psi^{\pm}(x,k)$ are $L^{16/9}(\mathbb{R}^2_x)$–valued continuous functions on $\{\, k \,\big|\, |k| \in (0,\infty)\backslash\sigma_p(H)\big\}$.*

As we mentioned in section 3, we shall use the integral operators T_1 and $T_{1/2}$; see (3.17). It will be convenient to split T_1 into two parts:

$$T_1 = T_{10} + T_{1\infty}, \tag{4.5}$$

where

$$\begin{aligned} T_{10}u(x) &= \int_{|x-y|\le 1} |x-y|^{-1} u(y)\, dy, \\ T_{1\infty}u(x) &= \int_{|x-y|> 1} |x-y|^{-1} u(y)\, dy. \end{aligned}$$

Then it follows from Lemma 3.3 and (4.2) that

$$\begin{aligned} &|\varphi^{\pm}(x,k)| \le 1 + C_{ab} \\ &\times \Big\{ \big|\big(T_{10}\psi^{\pm}(\cdot,\, k)\big)(x)\big| + \big|\big(T_{1\infty}\psi^{\pm}(\cdot,\, k)\big)(x)\big| + \big(T_{1/2}|\psi^{\pm}(\cdot,\, k)|\big)(x) \Big\} \end{aligned} \tag{4.6}$$

for all $(x,\, k) \in \mathbb{R}^2 \times \{\, k \,|\, a \le |k| \le b\big\}$, where C_{ab} is a positive constant.

Lemma 4.5 *If $16/9 \le q < 16$, then $T_{10}\psi^{\pm}(\cdot,\, k) \in L^q(\mathbb{R}^2)$. Moreover, there exits a positive constant C_{ab} such that*

$$\|T_{10}\psi^{\pm}(\cdot,\, k)\|_{L^q} \le C_{ab}$$

for all $k \in \{\, k \,|\, a \le |k| \le b\big\}$.

Proof. We write

$$\big(T_{10}\psi^{\pm}(\cdot,k)\big)(x) = \int_{\mathbb{R}^2} f_0(x-y)\psi^{\pm}(y,k)dy, \quad f_0(x) := |x|^{-1}\chi_0(x),$$

where $\chi_0(x)$ is the characteristic function for the unit disk $\{\, x \,|\, |x| \le 1\}$. It is easy to see that

$$f_0 \in L^p(\mathbb{R}^2_x) \text{ for all } p \in (0,\, 2). \tag{4.7}$$

Using Lemma 4.4 and the Young inequality (cf. Lemma A.2 in the appendix) with $r = 16/9$, we get

$$\|T_{10}\psi^{\pm}(\cdot,k)\|_{L^q} \le \|f_0\|_{L^p}\|\psi^{\pm}(\cdot,k)\|_{L^{16/9}}$$

for $\frac{1}{q} = \frac{1}{p} + \frac{9}{16} - 1$ $(1 \le p, q \le \infty)$. Noticing (4.7), we have

$$\frac{1}{16} < \frac{1}{q} \le \frac{9}{16} \Longleftrightarrow \frac{16}{9} \le q < 16.$$

Thus we get the lemma. $\square$

We are now in a position to prove the main theorem in this section, namely Theorem 4.1. In the proof below, we shall apply Lemma 4.5 with $q = 3$.

Proof of Theorem 4.1 Let $1/2 < s < \sigma - 1$. Noticing the definition (3.17) and the Schwarz inequality, we have

$$\big(T_{1/2}|\psi^{\pm}(\cdot, k)|\big)(x) \le \left\{ \int_{\mathbb{R}^2} \frac{1}{|x-y|\langle y\rangle^{2s}} dy \right\}^{1/2} \left\{ \int_{\mathbb{R}^2} \langle y\rangle^{2s} |\psi^{\pm}(y,k)|^2 dy \right\}^{1/2}.$$

Using Lemma 4.2 and Lemma A.1 in the appendix with $\beta = 1,\ \gamma = 2s > 1,\ n = 2$, we get

$$|\big(T_{1/2}|\psi^{\pm}(\cdot, k)|\big)(x)| \le C'_{ab1} \tag{4.8}$$

for all $(x,\ k) \in \mathbb{R}^2 \times \big\{\, k \,|\, a \le |k| \le b \big\}$, where C'_{ab1} is a positive constant.

Lemma 4.4, together with the Hölder inequality, yields

$$\begin{aligned} |T_{1\infty}\psi^{\pm}(x,k)| \le & \left\{ \int_{|x-y|>1} |x-y|^{-16/7}\, dy \right\}^{7/16} \\ & \times \left\{ \int_{\mathbb{R}^2} |\psi^{\pm}(y,k)|^{16/9}\, dy \right\}^{9/16} \\ \le & C'_{ab2} \end{aligned} \tag{4.9}$$

for all $(x,\ k) \in \mathbb{R}^2 \times \big\{\, k \,|\, a \le |k| \le b \big\}$, where C'_{ab2} is a positive constant.

Combining (4.8), (4.9) and (4.6), we have thus shown that

$$\begin{aligned} |\varphi^{\pm}(x,\ k)| \le & 1 + C_{ab}\big\{C'_{ab1} + C'_{ab2} + \big(T_{10}|\psi^{\pm}(\cdot, k)|\big)(x)\big\} \\ = & C''_{ab}\big\{1 + \big(T_{10}|V(\cdot)\varphi^{\pm}(\cdot, k)|\big)(x)\big\}. \end{aligned} \tag{4.10}$$

(Recall (4.4).) Here we would like to utilize the fact that T_{10} is positivity preserving, i.e.

$$T_{10}u \ge 0 \text{ if } u \ge 0. \tag{4.11}$$

It then follows from (4.10) and (4.11) that

$$\begin{aligned} |\varphi^{\pm}(x,\ k)| \le & C''_{ab}\Big\{1 + \Big(T_{10}|V(\cdot)|C''_{ab}\big\{1 + \big(T_{10}|V(\cdot)\varphi^{\pm}(\cdot, k)|\big)\big\}\Big)(x)\Big\} \\ = & C''_{ab}\Big\{1 + C''_{ab}\big(T_{10}|V(\cdot)|\big)(x) \\ & + C''_{ab}\Big(T_{10}|V(\cdot)|\big(T_{10}|\psi^{\pm}(\cdot, k)|\big)\Big)(x)\Big\} \end{aligned} \tag{4.12}$$

(Again recall (4.4).)

With the same notation as in the proof of Lemma 4.5, we have

$$0 \le \big(T_{10}|V(\cdot)|\big)(x) \le \|f_0\|_{L^{3/2}}\|V\|_{L^3} < +\infty, \tag{4.13}$$

where we have used the Hölder inequality.

Similarly, by using the Hölder inequality and applying Lemma 4.5, we have

$$\begin{aligned} 0 \le & T_{10}|V(\cdot)|\big(T_{10}|\psi^{\pm}(\cdot, k)|\big)\Big)(x) \\ \le & \|f_0\|_{L^{3/2}}\|V\|_{L^\infty}\|T_{10}|\psi^{\pm}(\cdot, k)|\|_{L^3} \\ \le & C'_{ab3} \end{aligned} \tag{4.14}$$

for all $(x, k) \in \mathbb{R}^2 \times \big\{ k \,|\, a \le |k| \le b \big\}$, where C'_{ab3} is a positive constant.

Combining (4.12) with (4.13) and (4.14), we obtain the desired conclusion. □

5 Generalized eigenfunction expansions

The task in this section is to establish the completeness of the generalized eigenfunction. The idea is the same as in our previous work [19]. For this reason, we shall only state the results and omit the proofs.

It is obvious that V is a bounded selfadjoint operator in $L^2(\mathbb{R}^2)$, and that $H = H_0 + V$ defines a selfadjoint operator in $L^2(\mathbb{R}^2)$, whose domain is $H^1(\mathbb{R}^2)$ (see [15, Theorem 5.8]). Moreover H is essentially selfadjoint on $C_0^\infty(\mathbb{R}^2)$ (see [15]). Since V is relatively compact with respect to H_0, it follows from [12, p.113, Corollary 2] that

$$\sigma_e(H) = \sigma_e(H_0) = [0, \infty).$$

The first result in this section is the asymptotic completeness of wave operators (cf. [19]).

Theorem 5.1 *Let H_0, H be defined by* (1.1) *and $V(x)$ satisfy* (1.2). *Then there exist the limits*

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0},$$

and the asymptotic completeness holds:

$$\mathcal{R}(W_\pm) = \mathcal{H}_{ac}(H),$$

where $\mathcal{H}_{ac}(H)$ denotes the absolutely continuous subspace for H.

We need to remark that $\sigma_p(H)\cap(0,\infty)$ is a discrete set. This fact was first proved by B. Simon [13, Theorem 2.1]. Moreover, he proved that each eigenvalue in the set $\sigma_p(H)\cap(0,\infty)$ has finite multiplicity. Finally, using Theorem 4.1 and Theorem 5.1, we can establish the eigenfunction expansion theorem as follows (see our previous work [19] for the details).

Theorem 5.2 *Let H_0, H be defined by* (1.1) *and $V(x)$ satisfy* (1.2). *Let $s>1$ and $[a,b]\subset(0,\infty)\backslash\sigma_p(H)$. For $u\in L^{2,s}(\mathbb{R}^2)$, let $\mathcal{F}_\pm$ be defined by*

$$\mathcal{F}_\pm u(k) := (2\pi)^{-1}\int_{\mathbb{R}^2} u(x)\overline{\varphi^\pm(x,k)}dx.$$

Then for any $f\in L^{2,s}(\mathbb{R}^2)$, we have

$$E_H([a,b])f(x) = (2\pi)^{-1}\int_{a\le|k|\le b}\mathcal{F}_\pm f(k)\varphi^\pm(x,k)dk,$$

where E_H is the spectral measure of H.

6 Asymptotic behaviors of the generalized eigenfunctions

We shall first show that the generalized eigenfunctions $\varphi^\pm(x,k)$, defined by (2.4), are distorted plane waves, and give estimates of the differences between $\varphi^\pm(x,k)$ and the plane wave $\varphi_0(x,k)=e^{ix\cdot k}$ (Theorem 6.1). We shall next prove that $\varphi^\pm(x,k)$ are asymptotically equal to the sums of the plane wave and the spherical waves $e^{\mp i|x||k|}/|x|^{1/2}$ under the assumption that $\sigma>2$, and shall give estimates of the differences between $\varphi^\pm(x,k)$ and the sums mentioned above (Theorem 6.2).

The similar estimates were discussed in T. Ikebe [5, §3] and our previous work [17, §10], though our arguments below are slightly different from those of [5] or [17], and our estimates are slight refinements of those of [5] or [17].

The main theorems in this section are

Theorem 6.1 *Let $\sigma>3/2$. If $|k|\in(0,+\infty)\backslash\sigma_p(H)$, then*

$$|\varphi^\pm(x,k)-\varphi_0(x,k)|\le C_k\begin{cases}\langle x\rangle^{-(\sigma-3/2)} & \text{if }\ 3/2<\sigma<2,\\ \langle x\rangle^{-1/2}\log(1+\langle x\rangle) & \text{if }\ \sigma=2,\\ \langle x\rangle^{-1/2} & \text{if }\ \sigma>2.\end{cases}$$

where the constant C_k is uniform for k in any compact subset of

$$\left\{k\middle||k|\in(0,+\infty)\backslash\sigma_p(H)\right\}.$$

Theorem 6.2 *Let $\sigma > 2$ and*

$$f^{\pm}(\lambda, \omega_x, \omega_k) := \left(\frac{\lambda}{\pi}\right)^{1/2} (1 \mp i) \int_{\mathbb{R}^2} e^{\pm i\lambda\omega_x \cdot y} V(y) \varphi^{\pm}(y, \lambda\omega_k) dy, \tag{6.1}$$

where $\omega_x = x/|x|$, $\omega_k = k/|k|$. Then for $|x| \geq 1$

$$\left| \varphi^{\pm}(x,k) - \left(\varphi_0(x,k) + \frac{e^{\mp i|k||x|}}{|x|^{1/2}} f^{\pm}(|k|, \omega_x, \omega_k) \right) \right|$$

$$\leq C_k \begin{cases} |x|^{-(\sigma-1)/2} & \text{if } \ 2 < \sigma < 3, \\ |x|^{-1} & \text{if } \ \sigma \geq 3, \end{cases} \tag{6.2}$$

where the constant C_k is uniform for k in any compact subset of

$$\left\{ k \middle| |k| \in (0, +\infty) \backslash \sigma_p(H) \right\}.$$

We should like to remark that what makes the discussions below possible is the estimate in Theorem 4.1.

Proof of Theorem 6.1. In view of (4.2), Theorem 3.2 and Theorem 4.1, it is clear that there is a positive constant C_k, which is uniform for k in any compact subset of $\{ k \,|\, |k| \in (0, +\infty) \backslash \sigma_p(H) \}$, such that

$$\begin{aligned} |\varphi^{\pm}(x,k) - \varphi_0(x,k)| \leq & C_k (T_1|V|)(x) + (T_{1/2}|V|)(x) \\ \leq & C_k \Big(\int_{\mathbb{R}^2} \frac{1}{|x-y|\langle y \rangle^{\sigma}} dy + \int_{\mathbb{R}^2} \frac{1}{|x-y|^{1/2}\langle y \rangle^{\sigma}} dy \Big). \end{aligned} \tag{6.3}$$

(Recall that T_1 and $T_{1/2}$ were introduced in (3.17).) We apply Lemma A.1 with $n = 2, \beta = 1, \gamma = \sigma > 3/2$, and get

$$\int_{\mathbb{R}^2} \frac{1}{|x-y|\langle y \rangle^{\sigma}} dy \leq C_{\sigma} \begin{cases} \langle x \rangle^{-(\sigma-1)} & \text{if } \ 3/2 < \sigma < 2, \\ \langle x \rangle^{-1} \log(1 + \langle x \rangle) & \text{if } \ \sigma = 2, \\ \langle x \rangle^{-1} & \text{if } \ \sigma > 2, \end{cases} \tag{6.4}$$

where C_{σ} is a constant depending only on σ. Similarly, we apply Lemma A.1 with $n = 2, \beta = 1/2, \gamma = \sigma > 3/2$, and get

$$\int_{\mathbb{R}^2} \frac{1}{|x-y|^{1/2}\langle y \rangle^{\sigma}} dy \leq C_{\sigma} \begin{cases} \langle x \rangle^{-(\sigma-3/2)} & \text{if } \ 3/2 < \sigma < 2, \\ \langle x \rangle^{-1/2} \log(1 + \langle x \rangle) & \text{if } \ \sigma = 2, \\ \langle x \rangle^{-1/2} & \text{if } \ \sigma > 2. \end{cases} \tag{6.5}$$

The theorem is a direct consequence of (6.3), (6.4) and (6.5). □

We shall give a proof of Theorem 6.2 by means of a series of lemmas.

Lemma 6.3 *Let $\sigma > 2$. If*

$$0 \le f(x) \le C|x|^{-1}, \tag{6.6}$$

then

$$\int_{\mathbb{R}^2} f(x-y)\langle y\rangle^{-\sigma} dy = O(|x|^{-1}) \tag{6.7}$$

as $|x| \to \infty$, where C is a constant.

Proof. Applying Lemma A.1 with $n = 2, \beta = 1, \gamma = \sigma > 2$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} f(x-y)\langle y\rangle^{-\sigma} dy \le & C \int_{\mathbb{R}^2} \frac{1}{|x-y|\langle y\rangle^{\sigma}} dy \\ \le & C'\langle x\rangle^{-1}, \end{aligned} \tag{6.8}$$

where C and C' are constants. It is apparent that (6.8) gives the lemma. □

In view of (3.11), (3.12) and (3.19), we get

$$\left| g_\lambda^{\pm}(x) - \left(\frac{\lambda}{\pi}\right)^{1/2} (1 \mp i) \frac{e^{\mp i\lambda|x|}}{|x|^{1/2}} \right| \le C|x|^{-1}, \tag{6.9}$$

where C is a constant. Then, using Lemma 6.3, (4.2) and (3.12), we see that

$$\begin{aligned} & \varphi^{\pm}(x,k) - \varphi_0(x,k) \\ = & \left(\frac{\lambda}{\pi}\right)^{1/2} (1 \mp i) \int_{\mathbb{R}^2} \frac{e^{\mp i|k||x-y|}}{|x-y|^{1/2}} V(y)\varphi^{\pm}(y,k)\, dy + O(|x|^{-1}) \end{aligned} \tag{6.10}$$

as $|x| \to \infty$. Now, noticing (6.1), (6.2) and (6.10), we need to consider the integral of the form

$$\int_{\mathbb{R}^2} \left\{ \frac{e^{ia|x-y|}}{|x-y|^{1/2}} - \frac{e^{ia(|x|-\omega_x\cdot y)}}{|x|^{1/2}} \right\} u(y) dy, \tag{6.11}$$

where $a \in \mathbb{R}$ and u is a function satisfying

$$|u(x)| \le C\langle x\rangle^{-\sigma}, \quad \sigma > 2. \tag{6.12}$$

Lemma 6.4 *Let u satisfy (6.12). Then for $|x| \ge 1$ we have*

$$\left| \int_{|y|\ge\sqrt{|x|}} \frac{e^{ia(|x|-\omega_x\cdot y)}}{|x|^{1/2}} u(y) dy \right| \le C_1 \|\langle\cdot\rangle^{\sigma} u\|_{L^\infty} |x|^{-(\sigma-1)/2}, \tag{6.13}$$

$$\left| \int_{|y|\ge\sqrt{|x|}} \frac{e^{ia|x-y|}}{|x-y|^{1/2}} u(y) dy \right| \le C_2 \|\langle\cdot\rangle^{\sigma} u\|_{L^\infty} |x|^{-(\sigma-1)/2}. \tag{6.14}$$

Proof. We obtain

$$\left|\int_{|y|\geq\sqrt{|x|}} e^{ia(|x|-\omega_x\cdot y)}u(y)dy\right| \leq C_1\|\langle\cdot\rangle^\sigma u\|_{L^\infty}|x|^{-(\sigma-2)/2} \tag{6.15}$$

by similar arguments in [17, (10.15)]. This inequality implies (6.13).

To prove (6.14), we write

$$F_0(x) := \left\{ y\in\mathbb{R}^2 \;\middle|\; |y|\geq\sqrt{|x|},\ |x-y|\leq\frac{|x|}{2}\right\}, \tag{6.16}$$

$$F_1(x) := \left\{ y\in\mathbb{R}^2 \;\middle|\; |y|\geq\sqrt{|x|},\ |x-y|\geq\frac{|x|}{2}\right\}, \tag{6.17}$$

and get

$$\left|\int_{F_0}\frac{e^{ia|x-y|}}{|x-y|^{1/2}}u(y)dy\right| \leq C'\|\langle\cdot\rangle^\sigma u\|_{L^\infty}|x|^{-(\sigma-3/2)}, \tag{6.18}$$

$$\left|\int_{F_1}\frac{e^{ia|x-y|}}{|x-y|^{1/2}}u(y)dy\right| \leq C''\|\langle\cdot\rangle^\sigma u\|_{L^\infty}|x|^{-(\sigma-1)/2}, \tag{6.19}$$

by similar arguments in [17, (10.17) and (10.18)].

Since $\sigma>2 \Leftrightarrow \sigma-3/2>(\sigma-1)/2$, we conclude from (6.18) and (6.19) that the inequality (6.14) holds. □

In view of (6.11) and Lemma 6.4, it is sufficient to evaluate the integral of the form

$$\int_{|y|\leq\sqrt{|x|}}\left\{\frac{e^{ia|x-y|}}{|x-y|^{1/2}}-\frac{e^{ia(|x|-\omega_x\cdot y)}}{|x|^{1/2}}\right\}u(y)dy. \tag{6.20}$$

We split it into two parts:

$$\begin{aligned}&\frac{1}{|x|^{1/2}}\int_{|y|\leq\sqrt{|x|}}\left\{e^{ia|x-y|}-e^{ia(|x|-\omega_x\cdot y)}\right\}u(y)dy\\ &+\int_{|y|\leq\sqrt{|x|}}e^{ia|x-y|}\left(\frac{1}{|x-y|^{1/2}}-\frac{1}{|x|^{1/2}}\right)u(y)dy.\end{aligned} \tag{6.21}$$

and evaluate these two integrals separately.

Lemma 6.5 *If $\sqrt{|x|}\geq 5$ and $|y|\leq\sqrt{|x|}$, then*

$$\Big||x-y|-(|x|-\omega_x\cdot y)\Big| \leq 3\sqrt{2}\frac{|y|^2}{|x|}. \tag{6.22}$$

For the proof of this lemma, see [17, (10.26)].

Lemma 6.6 *Under the same assumptions as in Lemma 6.4, we have*

$$\left|\frac{1}{|x|^{1/2}}\int_{|y|\le\sqrt{|x|}}\left\{e^{ia|x-y|}-e^{ia(|x|-\omega_x\cdot y)}\right\}u(y)dy\right|$$
$$\le C_3|a|\ \|\langle\cdot\rangle^\sigma u\|_{L^\infty}\begin{cases}|x|^{-(\sigma-1)/2} & \text{if } 2<\sigma<4,\\ |x|^{-3/2}\log(1+|x|) & \text{if } \sigma=4,\\ |x|^{-3/2} & \text{if } \sigma>4.\end{cases} \tag{6.23}$$

for $\sqrt{|x|}\ge 5$.

Proof. Let $\sqrt{|x|}\ge 5$. In a similar fashion to in [17, (10.28) and (10.30)], we get

$$\left|\frac{1}{|x|^{1/2}}\int_{|y|\le\sqrt{|x|}}\left\{e^{ia|x-y|}-e^{ia(|x|-\omega_x\cdot y)}\right\}u(y)dy\right|$$
$$\le 3\sqrt{2}|a|\ \|\langle\cdot\rangle^\sigma u\|_{L^\infty}\frac{1}{|x|^{3/2}}\left|\int_{|y|\le\sqrt{|x|}}|y|^2\langle y\rangle^{-\sigma}dy\right| \tag{6.24}$$

and

$$\begin{aligned}\left|\int_{|y|\le\sqrt{|x|}}|y|^2\langle y\rangle^{-\sigma}dy\right| &\le 2^{\sigma/2}\int_0^{\sqrt{|x|}}(1+r)^{-\sigma+3}dr\\ &\le \begin{cases}2^{\sigma/2}\dfrac{|x|^{-(\sigma-4)/2}}{4-\sigma} & \text{if } 2<\sigma<4,\\ 2^{\sigma/2}\log(1+|x|) & \text{if } \sigma=4,\\ 2^{\sigma/2}\dfrac{1}{4-\sigma} & \text{if } \sigma>4.\end{cases}\end{aligned} \tag{6.25}$$

Combining (6.24) with (6.25) yields the desired inequalities. □

Lemma 6.7 *Under the same assumptions as in Lemma 6.4, we have*

$$\left|\int_{|y|\le\sqrt{|x|}}e^{ia|x-y|}\left(\frac{1}{|x-y|^{1/2}}-\frac{1}{|x|^{1/2}}\right)u(y)dy\right|$$
$$\le\quad C\,\|\langle\cdot\rangle^\sigma u\|_{L^\infty}\begin{cases}|x|^{-\sigma/2} & \text{if } 2<\sigma<3,\\ |x|^{-3/2}\log(1+|x|) & \text{if } \sigma=3,\\ |x|^{-3/2} & \text{if } \sigma>3.\end{cases}\quad . \tag{6.26}$$

for $\sqrt{|x|}\ge 5$.

Proof. It is follows that

$$\left|\frac{1}{|x|^{1/2}}-\frac{1}{|x-y|^{1/2}}\right| = \frac{\big||x-y|^{1/2}-|x|^{1/2}\big|}{|x|^{1/2}|x-y|^{1/2}} = \frac{||x-y|-|x||}{|x|^{1/2}|x-y|^{1/2}\,\big||x-y|^{1/2}+|x|^{1/2}\big|}. \tag{6.27}$$

If $\sqrt{|x|}\geq 5$ and $|y|\leq\sqrt{|x|}$, then Lemma 6.5 implies

$$\big||x-y|-|x|\big|\leq |y|+3\sqrt{2}\frac{|y|^2}{|x|}. \tag{6.28}$$

If $\sqrt{|x|}\geq 5$ and $|y|\leq\sqrt{|x|}$, we then have

$$|x-y|\geq |x|-|y|\geq |x|-\frac{|x|}{5}=\frac{4}{5}|x|. \tag{6.29}$$

Hence, it follows from (6.27), (6.28) and (6.29) that

$$\left|\frac{1}{|x|^{1/2}}-\frac{1}{|x-y|^{1/2}}\right|\leq C'\frac{|y|}{|x|^{3/2}}+C''\frac{|y|^2}{|x|^{5/2}}. \tag{6.30}$$

when $\sqrt{|x|}\geq 5$ and $|y|\leq\sqrt{|x|}$. Using this inequality, we arrive at

$$\begin{aligned} &\left|\int_{|y|\leq\sqrt{|x|}} e^{ia|x-y|}\left(\frac{1}{|x-y|^{1/2}}-\frac{1}{|x|^{1/2}}\right)u(y)dy\right| \\ \leq\ & C'\|\langle\cdot\rangle^{\sigma}u\|_{L^\infty}\frac{1}{|x|^{3/2}}\int_{|y|\leq\sqrt{|x|}}|y|\langle y\rangle^{-\sigma}dy \\ & +C''\|\langle\cdot\rangle^{\sigma}u\|_{L^\infty}\frac{1}{|x|^{5/2}}\int_{|y|\leq\sqrt{|x|}}|y|^2\langle y\rangle^{-\sigma}dy. \end{aligned} \tag{6.31}$$

provided that $\sqrt{|x|}\geq 5$. Also we have

$$\left|\int_{|y|\leq\sqrt{|x|}}|y|\langle y\rangle^{-\sigma}dy\right|\leq\begin{cases} 2^{\sigma/2}\dfrac{|x|^{-(\sigma-3)/2}}{3-\sigma} & 2<\sigma<3, \\ 2^{\sigma/2}\log(1+|x|) & \sigma=3, \\ 2^{\sigma/2}\dfrac{1}{3-\sigma} & \sigma>3. \end{cases} \tag{6.32}$$

Combining (6.31) with (6.32) and (6.25), we conclude that the desired inequalities are verified. □

Proof of Theorem 6.2. Combining Lemmas 6.4, 6.6 and 6.7, we get for $|x| \geq 1$

$$\int_{\mathbb{R}^2} \left\{ \frac{e^{ia|x-y|}}{|x-y|^{1/2}} - \frac{e^{ia(|x|-\omega_x \cdot y)}}{|x|^{1/2}} \right\} u(y)dy$$

$$\leq C \begin{cases} |x|^{-(\sigma-1)/2} & \text{if } \ 2 < \sigma < 4, \\ |x|^{-3/2} \log(1+|x|) & \text{if } \ \sigma = 4, \\ |x|^{-3/2} & \text{if } \ \sigma > 4, \end{cases} \tag{6.33}$$

where C is a positive constant independent of a. This fact, together with (6.10) and (6.1), gives Theorem 6.2. □

Appendix

A Some inequalities

Lemma A.1 *Let $n \in \mathbb{N}$ and $\Phi(x)$ be defined by*

$$\Phi(x) := \int_{\mathbb{R}^n} \frac{1}{|x-y|^{\beta} \langle y \rangle^{\gamma}} dy.$$

If $0 < \beta < n$ and $\beta + \gamma > n$, then $\Phi(x)$ is a bounded continuous function satisfying

$$|\Phi(x)| \leq C_{\beta\gamma n} \begin{cases} \langle x \rangle^{-(\beta+\gamma-n)} & if \ \ 0 < \gamma < n, \\ \langle x \rangle^{-\beta} \log(1+\langle x \rangle) & if \ \ \gamma = n, \\ \langle x \rangle^{-\beta} & if \ \ \gamma > n. \end{cases}$$

where $C_{\beta\gamma n}$ is a constant depending on β, γ and n.

For the proof of this lemma, see [17, Lemma A.1].

Young's inequality for convolutions is as follows (cf. [14, P271]):

Lemma A.2 *Let $h = f * g$, then*

$$\|h\|_{L^q} \leq \|f\|_{L^p} \|g\|_{L^r}$$

where $1 \leq p, q, r \leq \infty$ and $1/q = 1/p + 1/r - 1$.

B Some special functions

For the reader's convenience, we summarize some properties of the Bessel function $J_0(\rho)$, the Neumann function $N_0(\rho)$ and the Struve function $H_0(\rho)$, whose definitions were given by (3.3), (3.5) and (3.4) respectively.

Lemma B.1 *Let $\rho \in \mathbb{R}$. Then*

$$J_0(\rho) = \left(\frac{2}{\pi\rho}\right)^{1/2} \cos\left(\rho - \frac{\pi}{4}\right) + O(\rho^{-3/2}) \tag{B.1}$$

$$N_0(\rho) = \left(\frac{2}{\pi\rho}\right)^{1/2} \sin\left(\rho - \frac{\pi}{4}\right) + O(\rho^{-3/2}) \tag{B.2}$$

as $\rho \to \infty$.

Proof. By [18, p. 199], we get

$$\begin{aligned} J_0(\rho) = \left(\frac{2}{\pi\rho}\right)^{1/2} & \Big[\cos(\rho - \frac{1}{4}\pi) \cdot \Big\{(0,0) + O(\rho^{-2})\Big\} \\ & - \sin(\rho - \frac{1}{4}\pi) \cdot \Big\{\frac{(0,1)}{2}\rho^{-1} + O(\rho^{-3})\Big\}\Big] \end{aligned}$$

as $\rho \to \infty$, where

$$(0,m) = \frac{\prod_{i=1}^{m}\{-(2i-1)^2\}}{m! \cdot 2^{2m}} = \frac{(-1)^m\{(2m-1)!!\}^2}{m! \cdot 2^{2m}}.$$

Noticing $(0,0) = 1, (0,1) = -1/4$, we have the asymptotic formula (B.1). Similarly, we have the asymptotic formula (B.2). □

Lemma B.2 *Let $\rho \in \mathbb{R}$. Then*

$$\mathbf{H}_0(\rho) = \left(\frac{2}{\pi\rho}\right)^{1/2} \sin\left(\rho - \frac{\pi}{4}\right) + O(\rho^{-1}) \tag{B.3}$$

as $\rho \to \infty$

Proof. Noting

$$\Gamma(k + \frac{3}{2}) = \sqrt{\pi}\frac{(2k+1)!!}{2^{k+1}},$$

we get the following formula from the definition (3.3).

$$\mathbf{H}_0(\rho) = \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k \rho^{2k+1}}{\{(2k+1)!!\}^2}$$

Then, by [18, p. 333], we get

$$\mathbf{H}_0(\rho) = N_0(\rho) + \frac{(\frac{1}{2}\rho)^{-1}}{\{\Gamma(1/2)\}^2}\sum_{k=0}^{p-1} \frac{(-1)^k(\frac{1}{2})_k(2k)!}{\rho^{2k} \cdot k!} + O(\rho^{-2p-1})$$

as $\rho \to \infty$, where,

$$(\frac{1}{2})_k = \frac{1}{2} \cdot \frac{3}{2} \cdot \dots \cdot \frac{2k-1}{2} = \frac{(2k-1)!!}{2^k}.$$

Since

$$\frac{(-1)^k(\frac{1}{2})_k(2k)!}{\rho^{2k}\cdot k!} = \frac{(-1)^k(2k-1)!!(2k)!}{\rho^{2k}2^k\cdot k!},$$

and

$$(2k)! = 2^k k!(2k-1)!!,$$

we get

$$\begin{aligned}\mathbf{H}_0(\rho) =& N_0(\rho) + \frac{2}{\pi}\sum_{k=0}^{p-1}(-1)^k\{(2k-1)!!\}^2\rho^{-2k-1} + O(\rho^{-2p-1}) \\ =& N_0(\rho) + O(\rho^{-1}).\end{aligned}$$

as $\rho \to \infty$. Finally, using Lemma B.1, we obtain this lemma. $\square$

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